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FORCED VIBRATIONS OF A PIEZOCERAMIC CYLINDRICAL SHELL WITH LONGITUDINAL POLARIZATION*

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Forced vibrations of a circular cylindrical piezoceramic shell with longitudinal polarization caused by an electric load applied to electrodes on the shell edge are considered. A numerical computation is performed by the partition method for the electroelastic state, and values of the coefficient of electromechanical coupling obtained by different formulas are compared.

1. We select the system of orthogonal curvilinear dimensionless coordinates ξ, φ such that the ξ-line coincides with the generatrix and the φ-line with the directrix of the cylinder.

We write down the system of equations for the electroelastic shell under consideration in the selected coordinates and we omit here certain equations not used below.

The equilibrium equations

$$dT_{1n}/d\xi - nS_{12n} + \lambda u_n = 0 \tag{1.1}$$

$$T_{2n} + \varepsilon^2 dN_{1n}/d\xi - \varepsilon^2 nN_{2n} + \lambda w_n = 0$$

$$dS_{12n}/d\xi + nT_{2n} - N_{2n} + \lambda v_n = 0 \tag{1.2}$$

$$N_{1n} = dG_{1n}/d\xi \tag{1.3}$$

The electroelasticity relationships

$$T_{1n} = \varepsilon_{1n} + \nu_2\varepsilon_{2n} - E_{1n}, \quad T_{2n} = \sigma(\varepsilon_{2n} + \nu_1\varepsilon_{1n}) - c_{12}E_{1n} \tag{1.4}$$

$$S_{12n} = S_{21n} = (\omega_n - d_{15}n_{22}c_2^{-1}E_{2n})/s_{44}{}^E n_{22} \tag{1.5}$$

$$G_{1n} = -\varepsilon^2\varkappa_{1n} \tag{1.6}$$

$$D_{1n} = \varepsilon_{33}{}^T (c_2d_{31})^{-1}E_{1n} + T_{2n} + d_{33}(d_{31})^{-1}T_{1n} \tag{1.7}$$

$$D_{2n} = \varepsilon_{11}{}^T (c_2d_{31})^{-1}E_{2n} + d_{15}d_{31}{}^{-1}S_{12n} \tag{1.8}$$

The electrostatics equations

$$dD_{1n}/d\xi - nD_{2n} = 0 \tag{1.9}$$

$$E_{1n} = -d\psi_n/d\xi, \quad E_{2n} = -n\psi_n \tag{1.10}$$

The strain-displacement formulas

$$\varepsilon_{1n} = du_n/d\xi, \quad \varepsilon_{2n} = -nv_n - qw_n \tag{1.11}$$

$$\omega_n = dv_n/d\xi + nu_n \tag{1.12}$$

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$$\varkappa_{1n} = d^2 w_n / d\xi^2 \tag{1.13}$$

The number q in the second formula in (1.11) will be zero or unity depending on the kind of shell vibrations that are taking place.

It is taken into account here and henceforth that the desired quantities vary with time t according to the law $e^{-i\omega t}$ (ω is the angular frequency), consequently, all the equations are written in the amplitude values of the desired quantities. Moreover, all the desired quantities. Moreover, all the desired quantities are expanded in Fourier series in the coordinate φ

$$P_1 = \sum_{n=1}^{\infty} P_{1n} \sin n\varphi, \quad P_2 = \sum_{n=1}^{\infty} P_{2n} \cos n\varphi \tag{1.14}$$

Any of the quantities $\psi, T_1, T_2, u, w, \varepsilon_1, \varepsilon_2, \varkappa_1, E_1, D_1, G_1$ is understood to be P_1 and any of the quantities $S_{12}, v, \omega, E_2, D_2, N_1; P_{1n}$ to be P_2 while P_{2n} are functions ot the variable ξ.

The u, v, w in (1.1)-(1.13) are displacements of points of the middle surface along the coordinates lines ξ, φ and in the normal direction, respectively, T_1, T_2 S_{12}, S_{21} are forces, G_1, G_2 are bending moments, N_1, N_2 are transverse forces, ψ is the electric potential, D_1, D_2 are the electric induction vector components, and E_1, E_2 are the electric field vector components. The standard notation /2/ is used for the physical constants, where the exceptions are certain quantities introduced in /1/ and given by the following formulas:

$$n_{11} = s_{33}{}^E/\delta, \; n_{22} = s_{11}{}^E/\delta, \; n_{12} = n_{21} = -s_{13}{}^E/\delta$$
$$c_1 = (d_{31}s_{33}{}^E - d_{33}s_{13}{}^E)/\delta, \; c_2 = (d_{33}s_{11}{}^E - d_{31}s_{13}{}^E)/\delta$$
$$\delta = s_{11}{}^E s_{33}{}^E - (s_{13}{}^E)^2, \; \nu_t = n_{12}/n_{ii}, \; \sigma = n_{11}/n_{22}, \; c_{12} = c_1/c_2$$

For convenience, the desired dimensionless quantities are introduced into (1.1)-(1.12) and are related to the desired dimensional quantities with the asterisk subscript in the following manner:

$$u = \frac{u_*}{R}, \quad v = \frac{v_*}{R}, \quad w = \frac{w_*}{R}, \quad T_i = \frac{T_{i*}}{2hn_{22}} \tag{1.15}$$
$$n_{22}\psi = c_2\psi_*, \quad D_i = \frac{D_{i*}}{d_{31}n_{22}}, \quad \varepsilon^2 N_i = \frac{N_{i*}}{2hn_{22}}$$
$$\varepsilon^2 G_i = \frac{G_{i*}}{2hRn_{22}}, \quad S_{12} = \frac{S_{12*}}{2hn_{22}}, \quad \lambda = \frac{\rho\omega^2 R^2}{n_{22}}, \quad \varepsilon^2 = \frac{h^3}{3R^2}$$

where ρ is the density of the piezoceramic, h is half the shell thickness, and R is its radius.

We will consider the shell edges covered by electrodes on which an electric potential is given that varies sinusoidally with time and generates forced shell vibrations.

One of the most important characteristics of the behaviour of piezoceramic elements is the electromechanical coupling coefficient (EMCC). It characterizes the ratio of the electrical (mechanical) energy stored in the body volume that is capable of inversion to all the mechanical (electrical) energy delivered from outside to the piezoceramic body /3/. Several methods exist for calculating the EMCC. One, proposed by Mason /2/ is to determine the EMCC k from the formula

$$k^2 = U_m{}^2/(U_e U_d) \tag{1.16}$$

For the problem under consideration the mutual energy U_m, the elastic energy U_e, and the electrical energy U_d are determined from the following formulas:

$$U_m = \int_S (d_{31}T_{2*}E_{1*} + d_{33}T_{1*}E_{1*} + d_{15}S_{12*}E_{2*})\, ds \tag{1.17}$$
$$U_e = \int_S (s_{11}{}^E T_{2*}^2 + 2s_{12}{}^E T_{1*}T_{2*} + s_{33}{}^E T_{1*}^2 + s_{44}{}^E S_{12*}^2)\, ds$$
$$U_d = \int_S (\varepsilon_{11}{}^T E_{2*}^2 + \varepsilon_{33}{}^T E_{1*}^2)\, ds$$

The integration here is over the shell middle surface S. The Mason formulas for the energies are given in three-dimensional form. Formulas (1.17) are obtained as a result of integrating the three-dimensional integrands over the thickness coordinate by discarding small components in the theory of shells. Moreover, the moments are discarded since they are small in the problem under consideration.

The dynamic EMCC is introduced by the formula

$$k_d{}^2 = (\omega_a{}^2 - \omega_r{}^2)/\omega_a{}^2 \tag{1.18}$$

where ω_r is the resonance frequency and ω_a is the antiresonance frequency.

A. F. Ulitko proposed another energetic EMCC formula /3/

$$k_u{}^2 = (U^{(p)} - U^{(k)})/U^{(p)} \tag{1.19}$$

$$U^{(p)}=\int_s\sum_{i=1}^{2}\left(\varepsilon_{i*}T_{i*}+\frac{1}{2}\omega_*S_{ij}+E_{i*}D_{i*}\right)dS\quad(j\neq i=1,2)$$

where $U^{(p)}$ is the internal energy found for an electroelastic body with disconnected electrodes, and $U^{(k)}$ is the internal energy of the same body with short-circuited electrodes. The formula for $U^{(k)}$ can be obtained by replacing the superscript (p) by (k) in the last relationship in (1.19). Exactly as in (1.17), the passage to the terminology of shell theory is performed in the formulas for $U^{(p)}$ and $U^{(k)}$.

The values of k_u^2 is calculated as follows /3/. Firstly the strains are determined from the solution of the original problem. Then two auxiliary problems should be solved which are the integration of a second order differential equation in the electric potential ψ to calculate $U^{(p)}$ and $U^{(k)}$. This equation can be obtained by substituting the relationships (1.7), (1.8), (1.10) into (1.9). It should here by considered that the strains are known from the solution of the original problem. The arbitraries of the integration for $\psi^{(p)}$ are determined from the integral condition on the surface of each disconnected electrode

$$\int_{S_e} D_1^{(p)}\,dS_e=0$$

(S_e is the electrode surface), and for $\psi^{(k)}$ from the following electrical conditions on the short-circuited electrodes:

$$\psi^{(k)}|_{\xi=\pm l}=0$$

where $2lR$ is the shell length.

After $\psi^{(p)}$ and $\psi^{(k)}$ have been found, it is not difficult to calculate $U^{(p)}$ and $U^{(k)}$.

2. Let the shell edges be covered by continuous electrodes on which values of the electric potential are given

$$\psi|_{\xi=\pm l}=\pm 1 \tag{2.1}$$

The problem under consideration is axisymmetric. It is described by the Eqs.(1.1), (1.3), (1.4), (1.6), (1.7), (1.9), (1.10), (1.11) and (1.13) in which we must put $n=0$, and moreover, we discard the subscript n on the desired quantities for brevity. After reduction, we write the system of equations in the form of three equations in the unknowns u, w, ψ:

$$\sum_{i=1}^{3}L_{ji}u_i=0\quad(j=1,2,3) \tag{2.2}$$

$$L_{11}=\frac{d^2}{d\xi^2}+\lambda,\quad L_{12}=-qv_2\frac{d}{d\xi},\quad L_{13}=\frac{d^2}{d\xi^2},\quad L_{21}=\sigma v_1\frac{d}{d\xi}$$

$$L_{22}=-\left[\varepsilon^2\frac{d^4}{d\xi^4}-(\lambda-q\sigma)\right],\quad L_{23}=c_{12}\frac{d}{d\xi},\quad L_{31}=a_1\frac{d^2}{d\xi^2}$$

$$L_{32}=-a_2q\frac{d}{d\xi},\quad L_{33}=a_3\frac{d^2}{d\xi^2}$$

$$a_1=\sigma v_1+\frac{d_{33}}{d_{31}},\quad a_2=\left(\sigma+v_2\frac{d_{33}}{d_{31}}\right),\quad a_3=-\frac{\varepsilon_{33}{}^T}{d_{31}c_1}+c_{12}+\frac{d_{33}}{d_{31}}$$

In (2.2) we have used the notation

$$u_1=u,\ u_2=w,\ u_3=\psi \tag{2.3}$$

We will now introduce a new unkown function Φ which we define from the equation

$$D\Phi=0,\ D=|L_{ij}|\ (i,j=1,2,3) \tag{2.4}$$

The desired quantities u_i are expressed in terms of Φ as follows:

$$u_i=D_{3i}\Phi \tag{2.5}$$

where D_{3i} is the cofactor of the element L_{3i}.

We will investigate the electroelastic state of the shell as a function of the frequency paraemter λ. An asymptotic analysis of the equations of forced vibrations, performed using the scheme of /1/, clarified the classification of the dynamic problems in which its own approximate system of equations corresponds to each kind of vibrations. Without going into the computations, which are similar to those made in /1/, we will describe the simplified problem.

To do this we divide the whole range of variation of the frequency parameter λ into the following four sections: 1) $0<\lambda<\lambda_0$, 2) $|\lambda-\lambda_0|\leqslant\varepsilon$, 3) $\lambda_0\ll\lambda<\lambda_1$, 4) $\lambda>\lambda_1$. The values of λ_0 and λ_1 will be determined below.

The complete problem is separated into a membrane principal problem (PP) and a simple edge effect in the first section. To obtain the PP equation, it is necessary to set $q = 1$, $\varepsilon = 0$ in (1.1) and (1.11). Equating ε to zero denotes discarding the moment terms. The complete problem is not separated into simpler ones in the second section since for $|\lambda - \lambda_0| \leqslant \varepsilon$ the solution of the additional problem (AP) ceases to be rapidly varying in the direction of the generatrix. The number λ_0 is determined from the condition that the coefficient of $w g_2^4$ equals zero in the resolving equation of the AP, presented in /1/

$$\lambda_0 = \frac{\sigma(1-\nu_1\nu_2)\varepsilon_{33}{}^T}{\varepsilon_{33}{}^T + d_{31}(\nu_2 c_2 - c_1)}$$

In this case the bending theory of shells should be used by setting q equal to one and ε different from zero.

In the third section of the change in the frequency parameter λ the complete problem is separated into the membrane PP and the AP, where the solution of the AP will be oscillating without damping since g_2^4 is negative in the resolving equation of the AP for values $\lambda_1 > \lambda \gg \lambda_0$. To obtain the equation of the membrane PP it is necessary to set $\varepsilon = 0$, $q = 1$ in (1.1) and (1.11).

For values of $\lambda > \lambda_1$ where $\lambda_1 \gg 1$ (the fourth section), the shell performs quasitangential vibrations for which the greatest value of w is λ times less than the greatest values of u. Consequently, the complete problem can be separated into a PP describing quasitangential vibrations and an AP whose solution is oscillation with index of variability $(1+s)/2$, where s is the index of variability of the solution of the PP, whose equations are obtained from (1.1) and (1.11) for $\varepsilon = 0$, $q = 0$. Here equating q to zero denotes neglecting w compared with u.

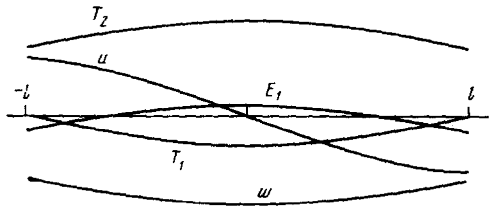

Fig.1

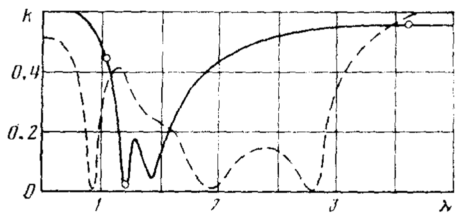

Fig.2

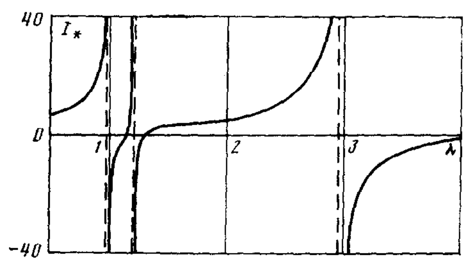

Fig.3

We will perform a computation using approximate equations as above. Such a separation of the electroelastic state for each kind of vibration simplifies the calculation considerably.

In those cases when the complete problem can be separated into a PP and an AP, we will confine ourselves to the solution of the PP since the greatest desired quantities of the electroelastic state (the stress, displacement and electric potential) described by the AP equations near the edge are R/h times less than the corresponding greatest values of the PP in the problem under consideration.

The calculation was performed for a shell made of the material PZT-4 /2/.

The amplitude values of the desired quantities of the first resonance are shown in Fig.1 for a shell of length $2R$ as a function of the longitudinal coordinate (the shell edges are free from mechanical supports).

The dependence of the different electromechanical coupling coefficients on the frequency parameter λ is shown in Fig.2: the curve of k in (1.16) is denoted by dashes, that of k_u in (1.19) by a solid line, and values of k_d in (1.18) are denoted by circles. Since k_d characterises the values of the EMCC within the interval $[\lambda_r, \lambda_a]$ the circle in Fig.2 is set in the middle of the interval (we consider k_d to be the arithmetic mean of the EMCC for λ_r and λ_a). As is seen from Fig.2, all values of k_d lie on the curve of k_u. Computation of the EMCC by

(1.16) yields qualitatively different results and since k_d is the characteristic confirmed experimentally and utilized extensively in practice, the agreement obtained confirms the validity of the energetic formula introduced by A.F. Ulitko.

The dependence of the current $I = I_* \omega S_\epsilon d_{31} n_{22}$ on the frequency parameter λ is shown in Fig.3. The EMCC k_d was determined according to the calculated values of the current (the resonance and antiresonance frequencies correspond to infinite and zeroth values of the current, respectively).

The results of the computation show that the EMCC takes the greatest values for quasi-tangential vibrations, where it decreases as the shell length increases. Values of the EMCC corresponding to the second section of the change in the frequency parameter λ are close to zero.

3. Let the shell edges be covered by slit electrodes on each of which the value of the electric potential is given. We will expand the electrical load in a Fourier series in the coordinate

$$\psi|_{\xi=\pm l} = \pm \sum_{n=1}^{\infty} t_n \sin n\varphi \tag{3.1}$$

where t_n are constants.

We take the membrane system of Eqs.(1.1), (1.2), (1.4), (1.5) and (1.7)-(1.12) as the initial system of equations by setting $\varepsilon = 0$. This means that vibrations for whose description the moment equations of the theory of piezoceramic shells should be used are eliminated from the consideration. There is a basis for this elimination: as is shown in the example of the axisymmetric problem, almost-zero values of the EMCC correspond to values of the frequency parameter λ for which the complete problem does not separate into PP and AP.

The action of the electrical load on the shell endfaces is similar to the action of a longitudinal load applied to the shell edge. Quasitransverse vibrations with low variability correspond to values of the frequency parameter λ comparable with one, and quasitangential vibrations correspond to the values $\lambda \gg 1$.

Quasitransverse vibrations with low variability are described by the membrane system of equations in which q should be considered to be equal to one. In the quasitangential vibrations equations q should be set equal to zero, which corresponds to discarding small deflections w as compared with the displacements u and v. The system of equations solved for u_n, v_n, ψ_n is written in the form of (2.2), while (2.3) must be replaced by the formulas

$$u_1 = u_n,\ u_2 = v_n,\ u_3 = \psi_n \tag{3.2}$$

The following formulas hold for the elements L_{ij}:

$$L_{11} = b_1 \frac{d^2}{d\xi^2} + b_2, \quad L_{12} = b_3 \frac{d}{d\xi}, \quad L_{13} = b_4 \frac{d^2}{d\xi^2} + b_5$$

$$L_{21} = e_1 \frac{d}{d\xi}, \quad L_{22} = e_2 \frac{d^2}{d\xi^2} + e_3, \quad L_{23} = e_4 \frac{d}{d\xi}$$

$$L_{31} = d_1 \frac{d^2}{d\xi^2} + d_2, \quad L_{31} = d_3 \frac{d}{d\xi}, \quad L_{23} = e_4 \frac{d}{d\xi}$$

$$b_1 = 1 - \frac{q}{q\sigma - \lambda} \sigma \nu_1 \nu_2, \quad b_2 = \lambda - \frac{n^2}{s_{44}{}^E n_{22}},$$

$$b_3 = n \left(\frac{q\sigma\nu_2}{\sigma q - \lambda} - \nu_2 - \frac{1}{s_{44}{}^E n_{22}} \right)$$

$$b_4 = 1 - \frac{q\nu_2 c_{12}}{\sigma q - \lambda}, \quad b_5 = -\frac{n^2 d_{15}}{s_{44}{}^E c_2}, \quad a = 1 - \frac{q\sigma}{q\sigma - \lambda}$$

$$e_1 = n \left[\frac{1}{s_{44}{}^E n_{22}} + \sigma \nu_1 a \right], \quad e_2 = \frac{1}{s_{44}{}^E n_{22}}, \quad e_3 = \lambda - n^2 \sigma a$$

$$e_4 = n \left[\frac{d_{15}}{s_{44}{}^E c_2} + c_{12} a \right], \quad d_1 = \sigma \nu_1 \frac{d_{31}}{\varepsilon_{33}{}^T} a + \frac{d_{33}}{\varepsilon_{33}{}^T} b_1$$

$$d_2 = -n^2 \frac{d_{15}}{\varepsilon_{33}{}^T s_{44}{}^E n_{22}}, \quad d_3 = -n \left[\left(\sigma \frac{d_{31}}{\varepsilon_{33}{}^T} + \nu_2 \frac{d_{33}}{\varepsilon_{33}{}^T} \right) a + \frac{d_{15}}{s_{44}{}^E \varepsilon_{33}{}^T n_{22}} \right]$$

$$d_4 = \left[-\frac{1}{c_2} + \frac{d_{31} c_{12}}{\varepsilon_{33}{}^T} a + \frac{d_{33}}{\varepsilon_{33}{}^T} b_4 \right], \quad d_5 = \frac{n^2}{c_2 \varepsilon_{33}{}^T} \left(\varepsilon_{11}{}^T - \frac{d_{15}{}^2}{s_{44}{}^E} \right)$$

The desired quantities u_i are expressed in terms of the auxiliary function Φ by (2.4) and (2.5).

On the shell edges conditions (3.1) should be satisfied:

$$\psi_n|_{\xi=\pm l} = \pm t_n \tag{3.3}$$

for the n-th term of the expansion of the electric potential ψ.

To be specific we will examine the case when there are two electrodes on each endface on which the following values of the electric potential are given:

$$\psi(\varphi)|_{\xi=\pm 1} = \begin{cases} \pm 1, & 0 < \varphi < \pi \\ \mp 1, & \pi < \varphi < 2\pi \end{cases} \tag{3.4}$$

We will consider the shell edges to be unclamped

$$T_1 = 0,\ S_{21} = 0\ (\xi = \pm l) \tag{3.5}$$

For shells with the boundary conditions (3.4) and (3.5) the first natural frequencies and the EMCC are calculated by means of (1.16), (1.18) and (1.19) as a function of the shell length.

The results of the computations are presented in the table. Its first column contains values of the ratio between the shell length and its radius, the second column is the resonance frequency, the third is the antiresonance frequency, and values of the EMCC calculated by different formulas are shown (increased by a factor of 10^3) in the following columns. The superscripts (r) and (a) denote the EMCC values $k^{(r)}$ and $k^{(a)}$ found for the resonance or antiresonance frequencies, respectively. As in the case of the axisymmetric problem, the values of k_d are in good agreement with the arithmetic mean of k_u in the interval $[\lambda_r, \lambda_a]$ (the agreement is even better with the mean-integral value of k_u).

l	λ_r	λ_a	$k_d\cdot 10^3$	$k_u^{(r)}\cdot 10^3$	$k_u^{(a)}\cdot 10^3$	$k^{(r)}\cdot 10^3$	$k^{(a)}\cdot 10^3$
0.5	2.07	2.24	282	370	206	60	275
	7.00	7.04	71	118	72	267	282
	9.86	10.80	310	401	200	268	359
	12.34	12.44	86	106	69	422	424
0.75	1.91	2.10	300	394	218	110	305
	4.63	4.76	166	238	108	278	319
	5.17	5.27	135	163	106	420	431
	7.87	8.88	336	402	270	154	399
1	1.64	1.82	312	410	232	265	388
	3.13	3.19	42	78	56	264	271
	4.48	4.53	110	168	60	471	475
	5.33	5.93	317	377	263	128	382
1.25	1.37	1.52	325	398	258	283	426
	2.68	2.70	86	89	74	229	230
	4.07	4.20	171	341	50	111	428
	4.26	4.53	250	265	235	502	453

Thus, a calculation has been performed for a cylindrical shell polarized in the longitudinal direction and with electrodes on the edges. It has been shown that either membrane equations should be used in the computation, or quasitangential vibrations equations for fairly high values of the frequency. The results of the computation enable us to deduce that the EMCC reaches the highest values for shells with free edges that perform vibrations for which the shell achieves the greatest displacements in the polarization direction. Moreover, the results of the computation confirm the efficiency of the method proposed by A.F. Ulitko for calculating the EMCC for any shell vibrations frequencies.

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